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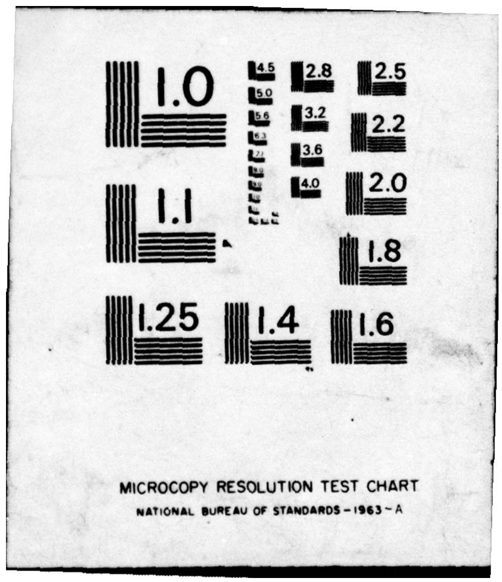
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THE STABILITY OF PLANE COUETTE FLOW

by

TERENCE COFFEE

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

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The effect of small disturbances on laminar plane Couette flow is studied. If the disturbances are infinitesimal, the effects are described by the Orr-Sommerfeld equation. Chebyshev polynomials are used to reduce the problem to linear algebra. The resulting eigenvalue problem is solved using the generalized Rayleigh quotient, which involves substantially less computer time than previous methods. Accurate eigenvalues are then computed for higher values of the parameters than has been done previously.

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

THE STABILITY OF PLANE COUETTE FLOW

by

Terence Coffee

Adviser: Professor Harry Rauch

The effect of small disturbances on laminar plane Couette flow is studied. If the disturbances are infinitesimal, the effects are described by the Orr-Sommerfeld equation. Chebyshev polynomials are used to reduce the problem to linear algebra. The resulting eigenvalue problem is solved using the generalized Rayleigh quotient, which involves substantially less computer time than previous methods. Accurate eigenvalues are then computed for higher values of the parameters than has been done previously. These values further confirm the belief that Couette flow is stable under infinitesimal disturbances.

Using the linear results as a starting point, the effect of finite disturbances is studied. A system of equations is derived from the Navier-Stokes equations, taking into account non-linear terms. In this case, the flow becomes turbulent for certain values of the parameters.

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1. INTRODUCTION

This paper is concerned with some results about the stability of plane Couette flow under small disturbances. In non-dimensional form, the problem consists of two dimensional fluid flow between two infinite planes, one at $y = 1$ moving at a speed of 1 to the right, and one at $y = -1$ moving at a speed of 1 to the left. If $u(x,y,t)$ is the horizontal component of the velocity of the flow, and $v(x,y,t)$ is the vertical component, then $u(x,y,t) = y$ and $v(x,y,t) = 0$ is a solution to the problem. This is the basic laminar flow. However, in practice laminar flow is hard to maintain, and most flows tend to become turbulent if disturbed. Knowledge of turbulent flow is in a very incomplete stage, with only some general ideas of the processes involved. Here we will be concerned with the somewhat simpler problem of transition; with the change from laminar to turbulent flow. This is the question of stability; that is, if the flow is disturbed, will it return to laminar flow (stability) or change to turbulent flow. Stability of course depends on the amplitude of the disturbance.

The first major contribution to the study of hydrodynamical stability can be found in the theoretical papers of Helmholtz, around 1868. Rayleigh and Reynolds

made further contributions near the end of the nineteenth century. In particular, dimensional analysis led Reynolds to the idea of what is now called the Reynolds number, a pure number inversely proportional to the viscosity. Experimental evidence led Reynolds to the conclusion that stability breaks down when this number exceeds a critical value.

Explaining this behavior proved difficult. The key equation was arrived at independently by Orr in 1907 and Sommerfeld in 1908, now called the Orr-Sommerfeld equation. While only valid for infinitesimal perturbations, it gave a starting point for explaining transitional behavior. This equation was applied to various types of fluid flows, including plane Couette flow, one of the simpliest cases.

The Orr-Sommerfled equation depends on two parameters; the Reynolds number R and the wavenumber α of the perturbation. The first approach to the equation was in terms of asymtotic analysis; in particular, picking α fixed and considering the limit as R approached infinity. Hopf made the first attempt in 1914, which was refined by Southwell and Chitty in 1930. For Couette flow, the first important work was done by Wasow in 1953 and Grohne in 1954, who developed similar asymtotic theories to show that the flow is stable for fixed α when R is sufficiently large. This has recently been refined by Davey in 1972,

who also showed that the flow is stable for fixed R when α is sufficiently large.

Numerical solution of the Orr-Sommerfeld equation has been difficult. The first solutions were obtained by Tollmien in 1929, who also obtained a critical Reynolds number. Lin in 1945 improved the mathematical procedures and laid the foundations for a general expansion of the stability analysis. However, fitting the general solutions to particular boundary solutions proved to be too mathematically complex.

Numerical investigation of Couette flow had to wait for the development of computers. Grohne in 1954 developed some results for the special case $\alpha = 1$. The first extensive numerical investigation was carried out by Gallagher and Mercer in 1962 for values of $\alpha R \leq 1000$, which confirmed the hypothesis that Couette flow is stable to infinitesimal perturbations. However, as αR gets larger, the problem becomes more and more singular, making the numerical solution more difficult.

Deardorff in 1963 extended slightly the parameter range covered by Gallagher and Mercer. More recently, Davey has computed solutions for αR up to 100,000. Using these results, Davey has detected a pattern in the solutions for the range that is not covered by the asymptotic analysis.

In this paper I will develop a more efficient method

of computing the solutions of the Orr-Sommerfeld equation, giving accurate solutions relatively easily for large values of αR. The results verify Davey's analysis.

The combination of numerical and asymtotic analysis leads to the conclusion that Couette flow is stable to infinitesimal disturbances for all values of the parameters. However, this is not borne out in practice. In a series or experiments Reichardt in 1956 was able to maintain laminar flow only for R up to about 750. The assumption is that non-linear effects cause the transition to turbulent flow.

Non-linear analysis is relatively recent, based mainly on the work of Meksyn, Stuart, and Watson, developed around 1960. But so far only fragmentary results have been developed, and there are essentially no such results for plane Couette flow. In the second part of this paper, some results of a non-linear analysis are discussed. In particular, if second order terms are considered, a transition from laminar to turbulent flow occurs. As in experiments, the size of the disturbance needed to produce turbulence depends upon the Reynolds number.

2. DERIVATION OF THE ORR-SOMMERFELD EQUATION

The general equations of a two-dimensional incompressible flow are the Navier-Stokes equations

(2.1) $u_t + uu_x + vu_y + p_x - (u_{xx} + u_{yy})/R = 0$

(2.2) $v_t + uv_x + vv_y + p_y - (v_{xx} + v_{yy})/R = 0$

(2.3) $u_x + v_y = 0$

where $u(x,y,t)$ is the fluid velocity parallel to the x-axis, $v(x,y,t)$ is the fluid velocity parallel to the y-axis, and $p(x,y,t)$ is the kinematic pressure. A derivation is given in Betchov and Criminale (1967). For plane Couette flow, the boundary conditions are

(2.4) $u(x,\pm 1,t) = \pm 1$

$v(x,\pm 1,t) = 0$

The basic laminar solution is

(2.5) $u(x,y,t) = y$

$v(x,y,t) = 0$

$p(x,y,t) = P(x)$

We now consider small perturbations of the laminar flow, so that

(2.6) $u(x,y,t) = y + u'(x,y,t)$

$v(x,y,t) = v'(x,y,t)$

$p(x,y,t) = P(x) + p'(x,y,t)$

where u', v', and p' are small.

Substituting into the Navier-Stokes equations, we obtain

$$(2.7)\quad u'_t + yu'_x + u'u'_x + v' + v'u'_y + P_x + p'_x$$
$$-(u'_{xx} + u'_{yy})/R = 0$$

$$(2.8)\quad v'_t + yv'_x + u'v'_x + v'v'_y + p'_y$$
$$-(v'_{xx} + v'_{yy})/R = 0$$

$$(2.9)\quad u'_x + v'_y = 0$$

We can eliminate the pressure term by differentiating (2.7) with respect to y, (2.8) with respect to x, and subtracting the equations. This results in

$$(2.10)\quad (u'_y - v'_x)_t + y(u'_{xy} - v'_{xx}) + u'_y u'_x + u'u'_{xy}$$
$$+v'_y + v'_y u'_y + v'u'_{yy} - u'_x v'_x - u'v'_{xx} - v'_x v'_y$$
$$-v'v'_{xy} - (u'_{yyy} + u'_{xxy} - v'_{xxx} - v'_{xyy})/R = 0$$

In the linear theory, we assume that the perturbations are small enough that we can ignore the second order terms. So (2.10) becomes

$$(2.11)\quad (u'_y - v'_x)_t + y(u'_y - v'_x)_x +$$
$$-(u'_{yyy} + u'_{xxy} - v'_{xxx} - v'_{xyy})/R = 0$$

Equation (2.9) permits the introduction of a stream function $\psi(x,y,t)$ such that $u' = \psi_y$ and $v' = -\psi_x$. Equation (2.9) is then satisfied automatically and (2.11) becomes

$$(2.12)\quad (\psi_{yy} + \psi_{xx})_t + y(\psi_{yy} + \psi_{xx})_x$$
$$-(\psi_{yyyy} + 2\psi_{xxyy} + \psi_{xxxx})/R = 0$$

We consider functions of the form $\psi(x,y,t) = \phi(y)\exp(i\alpha(x - ct))$, that is, periodic solutions in x and t, where α is the wavenumber of the

horizontal motion and c is an eigenvalue to be determined. Since we are assuming linearity, a series of solutions of this type will not interact with each other.

Equation (2.12) becomes

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi - i\alpha R(y - c)(\phi'' - \alpha^2\phi) = 0 \tag{2.13}$$

This is the well known Orr-Sommerfeld equation. The boundary conditions (2.4) become

$$\phi(\pm 1) = \phi'(\pm 1) = 0 \tag{2.14}$$

In this formulation, $\psi(x,y,t)$ is a complex valued function. But since the oringinal Navier-Stokes equations contain no complex quantities, $\frac{1}{2}(\psi(x,y,t) + \tilde{\psi}(x,y,t))$ is also a solution, and is real valued.

The problem has been reduced to a fourth order ordinary differential eigenvalue equation. The imaginary part of c, where $c = c_r + ic_i$, becomes the relevant result. If $c_i < 0$, the disturbance decays with time. So the flow is stable with respect to infinitesimal disturbances. If $c_i > 0$, the perturbations grow indefinitely with time. Of course once they become of any significant size the linear theory no longer applies.

3. METHOD OF SOLUTION

Various methods have been used to solve the Orr-Sommerfeld equation, generally some kind of finite-difference approximation. However, more efficiency can be obtained using Chebyshev polynomials.

Other sets of orthogonal polynomials have also been used in stability problems, usually chosen to satisfy the boundary conditions and have some relation to the Orr-Sommerfeld equation. The Chandrasekhar-Reid functions are the most common, though other types have been proposed. The advantages of the Chebyshev polynomials are discussed in general by Fox and Parker (1972) and in particular by Orszag (1971), who used them for the similar problem of plane Poiseuille flow.

Briefly, Chebyshev polynomials are particularly useful for obtaining accurate solutions. The Chebyshev polynomial approximations are of infinite order, in the sense that errors decrease more rapidly than any power of 1/N as N approaches infinity, where N is the number of terms used in the expansion. Hence, while other methods may give better answers for small N, accurate answers can be obtained much more rapidly using Chebyshev polynomials. Another advantage is the efficiency with which the coeffi-

cients may be determined from the differential equation. This makes the process of changing to a system of linear equations much less complicated.

So we will represent $\phi(y) = \sum_{n=0}^{N} a_n T_n(y)$, where $T_n(y)$ is the n'th degree Chebyshev polynomial of the first kind, defined by $T_n(\cos\theta) = \cos n\theta$. Derivatives and integrals of $\phi(y)$ can be represented relatively easily by using the relation

$$(3.1) \quad c_n T'_{n+1}(y)/(n+1) - d_{n-2} T'_{n-1}(y)/(n-1) = 2T_n(y)$$

where

$$c_n = \begin{cases} 0 & \text{if } n < 0 \\ 2 & \text{if } n = 0 \\ 1 & \text{if } n > 0 \end{cases}$$

and

$$d_n = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0 \end{cases}$$

The Orr-Sommerfeld equation can now be approximated by a series of linear equations in the a_n by substituting in the expansion for $\phi(y)$ and equating the coefficients of $T_n(y)$. This procedure, used by Orszag, has certain computational disadvantages, in that the matrices produced have coefficients of widely different orders of magnitude. This is due to the infinite sequence needed in working with derivatives.

For example, if $\phi'(y) = \sum_{n=0}^{N} b_n T_n(y) = \sum_{n=0}^{N} a_n T'_n(y)$ then $c_n b_n = 2 \sum_{p=n+1}^{\infty}{}^{2}\, p a_p$, where $\sum^{2}$ indicates that the summation is in jumps of two. However, the indefinite integral has a much simpler form. That is, if

$$\int \phi(y) = \sum_{n=0}^{N} a_n \int T_n(y) = \sum_{n=0}^{N+1} a^1{}_n T_n(y)$$

represents the indefinite integral of $\phi(y)$, we can find that

$$\sum_{n=0}^{N} a_n \int T_n(y) = \sum_{n=0}^{N} \frac{a_n}{2}\left[\frac{c_n T_{n+1}(y)}{n+1} - \frac{d_{n-2}\, T_{n-1}(y)}{n-1}\right]$$

$$= \sum_{n=0}^{N+1} \frac{a_{n-1}c_{n-1} - a_{n+1} d_{n-1}}{2n} T_n(y)$$

where $a_{N+1} = 0$, so that

$$(3.2) \quad a^1{}_n = \frac{a_{n-1}c_{n-1} - a_{n+1}}{2n} \qquad n = 1,2,\ldots,N+1$$

($a^1{}_0$ is an arbitrary constant of integration)

This is quicker to compute, and also leads to a matrix whose coefficients are much closer in size.

Writing (2.13) as

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi -$$
$$i\alpha R(y\phi'' - \alpha^2 y\phi - c\phi'' + \alpha^2 c\phi) = 0$$

and integrating four times, we obtain

$$(3.3) \quad \phi - 2\alpha^2\iint\phi + \alpha^4\iiiint\phi$$
$$-i\alpha R\left(\iint y\phi - 2\iiint\phi - \alpha^2\iiiint y\phi - c\iint\phi + \alpha^2 c\iiiint\phi\right) = 0$$

Letting $\iint\phi(y) = \sum_{n=0}^{N+2} a^2{}_n T_n(y)$

$\iiint\phi(y) = \sum_{n=0}^{N+3} a^3{}_n T_n(y)$

and $\iiiint\phi(y) = \sum_{n=0}^{N+4} a^4{}_n T_n(y)$

we can obtain, as above

$$(3.4) \quad a^1{}_n = \sum_{h=1}^{2} \frac{i^{h-1}\, c_{n-2+h}\, a_{n-2+h}}{2n}$$

$n = 1,2,\ldots$

(3.5) $a^2_n = \sum_{k=1}^{3}{}^{2} \sum_{h=1}^{3}{}^{2} \dfrac{i^{k+h-2}\, a_{n-4+k+h}}{4n(n-2+k)}$

$n = 2, 3, \ldots$

(3.6) $a^3_n = \sum_{l=1}^{3}{}^{2} \sum_{k=1}^{3}{}^{2} \sum_{h=1}^{3}{}^{2} \dfrac{i^{k+h+l+1}\, a_{n-6+k+h+l}}{8n(n-2+l)(n-4+k+l)}$

$n = 3, 4, \ldots$

(3.7) $a^4_n = \sum_{m=1}^{3}{}^{2} \sum_{l=1}^{3}{}^{2} \sum_{k=1}^{3}{}^{2} \sum_{h=1}^{3}{}^{2} \dfrac{i^{k+h+l+m}\, a_{n-8+k+h+l+m}\, c_{n-8+k+h+l+m}}{16n(n-2+m)(n-4+m+l)(n-6+m+l+k)}$

$n = 4, 5, \ldots$

If we set $y\phi(y) = \sum_{n=0}^{N+1} b_n T_n(y)$

$\iint y\phi(y) = \sum_{n=0}^{N+3} b^2_n T_n(y)$

and $\iiiint y\phi(y) = \sum_{n=0}^{N+5} b^4_n T_n(y)$

and use the relation

(3.8) $yT_n(y) = \frac{1}{2}(d_{n-1}T_{n-1}(y) + c_n T_{n+1}(y))$

we can obtain

(3.9) $b_n = \sum_{j=1}^{3}{}^{2} \dfrac{c_{n-2+j}\, a_{n-2+j}}{2}$

$n = 0, 1, 2, \ldots$

(3.10) $b^2_n = \sum_{j=1}^{3}{}^{2} \sum_{k=1}^{3}{}^{2} \sum_{h=1}^{3}{}^{2} \dfrac{i^{h+k-2}\, a_{n-6+k+h+j}\, c_{n-6+k+h+j}}{8n(n-2+k)}$

$n = 2, 3, \ldots$

(3.11) $b^4_n = \sum_{j=1}^{3}{}^{2} \sum_{m=1}^{3}{}^{2} \sum_{l=1}^{3}{}^{2} \sum_{k=1}^{3}{}^{2} \sum_{h=1}^{3}{}^{2} \dfrac{i^{m+l+k+h}\, a_{n-10+m+l+k+h+j}\, c_{n-10+m+l+k+h+j}}{32n(n-2+m)(n-4+m+l)(n-6+m+l+k)}$

$n = 4, 5, \ldots$

(Exception: if $n = 4$, $j = 3$, and $m = l = k = h = 1$, the above coefficient is doubled)

Substituting all of these expansions into (3.3), equating coefficients of $T_n(y)$, and multiplying by i, we obtain the system of equations

$$(3.12) \quad ia_n - 2i\alpha^2 a_n^2 + i\alpha^4 a_n^4 + \alpha Rb_n^2 - 2\alpha Ra_n^3$$

$$-\alpha^3 Rb^4{}_n - c\alpha Ra_n^2 + c\alpha^2 Ra^4{}_n = 0$$

$$n = 4,5,\ldots$$

Because of the integration, this equation for the cases $n = 0,1,2,3$ contains arbitrary constants. Hence we replace these equations by equations representing the boundary conditions. These conditions can be simplified using the relations

$$T_n(\pm 1) = (\pm 1)^n$$

and $\quad T'_n(\pm 1) = n^2(\pm 1)^{n-1}$

thus $\quad \phi(1) = \sum_{m=0}^{N} a_n = 0$

$$\phi(-1) = \sum_{m=0}^{N} (-1)^n a_n = 0$$

$$\phi'(1) = \sum_{m=0}^{N} n^2 a_n = 0$$

$$\phi'(-1) = \sum_{m=0}^{N} n^2(-1)^{n-1} a_n = 0$$

By adding and subtracting equations, we obtain the slightly simpler form

$$(3.13) \quad \sum_{n=0}^{N+2} a_n = 0$$

$$\sum_{n=1}^{N+2} a_n = 0$$

$$\sum_{n=0}^{N+2} n^2 a_n = 0$$

$$\sum_{n=1}^{N+2} n^2 a_n = 0$$

Using these four equations plus equation (3.12) for $n = 4,5,\ldots N$, we obtain a system of N+1 equations in N+1 unknowns. In matrix form this can be written as $XA - cYA = 0$, where X is a complex valued matrix, Y is a real valued matrix, and $A = (a_0, a_1, \ldots a_N)$ is the vector to be determined. So the oringinal differential equation is reduced to the above algebraic eigenvalue problem.

4. SOLVING THE MATRIX EQUATION: THE LR ALGORITHM

There are a number of standard methods for solving an algebraic eigenvalue problem. Wilkinson (1965) contains a fairly complete description. However, the present problem presents several difficulties.

First, the usual form for an algebraic eigenvalue problem is $XA - cA = 0$. An equation of the form $XA - cYA = 0$ is normally rewritten as $Y^{-1}XA - cA = 0$. Unfortunately, in our problem Y is not invertible. The first four rows of Y consist solely of zeros, since the eigenvalue c does not enter into the boundary conditions.

A method has been devised by Gary and Helgason (1970) to get around this difficulty. A sequence of elementary column transformations is used to put the first four rows of X into lower triangular form. This is accomplished by adding a multiple of the first column to the succeeding columns so that $x_{12}, x_{13}, \ldots$ all become zero. We continue the process for the second, third, and fourth columns, permuting the columns if necessary to put the largest element in the appropriate column. Since the elements in these rows are known, this is easy to program efficiently. The same column transformations are performed on Y. The first four rows of Y remain zero, so if we drop the

first four rows and columns of both X and Y, the new system has the same eigenvalues as the oringinal system. For convenience, this will still be denoted $XA - cYA = 0$, but Y is now generally non-singular.

To change this into the form $Y^{-1}XA - cA = 0$, programs from Forsythe and Moler (1967) were adapted. The matrix Y is first scaled, so the norms of the rows are approximately the same size. Then, using partial pivoting to reduce round off error, Y is decomposed into the form $Y = LU$, where L is a lower triangular matrix and U is an upper triangular matrix. Instead of actually interchanging the rows, a vector is introduced to keep track of which elements have been used as pivots, saving computer time. Then for each column X_i of X, the equation $YZ_i = X_i$ can be easily solved, where Z_i is the i'th column of $Y^{-1}X$. To insure accuracy, iterative improvement is used. That is, we form a residual $R_i = X_i - YZ_i$ and solve the system $YZ'_i = R_i$. Then $Z + Z'$ is a more accurate solution. Since most of the arithmetic in this procedure is in the oringinal decomposition, this is inexpensive. Again, for convenience, this new system can be denoted $XA - cA = 0$.

A second difficulty is that X is a general complex matrix. The most common eigenvalue problem is for a real symmetric matrix, and many of the standard procedures do not apply to complex matrices. Wildinson discusses only one method for a general complex matrix, the LR

algorithm, which is relatively complicated. To implement this, programs from Wilkinson and Reinsch (1971) were translated from Algol and adapted to this particular problem.

To reduce round off errors, the matrix is first balanced. Since errors in eigenvalue procedures depend on the size of the norm of the matrix, diagonal similarity transformations are used to reduce the norm of X. An iterative process replaces X by $D^{-1}XD$, until the norm of the i'th row and column of X are of the same order of magnitude.

Next the matrix is tranformed to upper Hessenberg form, that is, $x_{ij} = 0$ if $i > j + 1$. While no finite sequence of similarity transformations can reduce a matrix to upper triangular form, elementary matrices can introduce zeros in all places below the subdiagonal of the main diagonal.

Finally, the LR algorithm is applied to the upper Hessenberg matrix. Again, a sequence of similarity transformations is used to reduce the elements under the main diagonal. Since only the elements in the subdiagonal are non-zero, the procedure converges much more rapidly than for a general matrix. As soon as the term x_{21} becomes negligible, x_{11} is an eigenvalue for the matrix. The first row and column can then be deleted and the procedure repeated.

There are some drawbacks to the above procedure. The process is long, being relatively expensive in computer time. Because of the efficiency of the approximation by Chebyshev polynomials, it does compare favorable with other methods, but it does compute a lot of information of no immediate value. In particular, in stability theory we are concerned with the eigenvalue having the largest imaginary part, since this will represent the solution closest to being unstable. But the LR algorithm computes all the eigenvalues. Since in some cases we will go up to $N = 80$, this would be 76 eigenvalues computed that we have no interest in.

So in the next chapter we devise an alternate method of solving the eigenvalue problem, one apparently not previously used in stability problems.

5. SOLVING THE MATRIX EQUATION: THE RAYLEIGH QUOTIENT

The second method of solution is based on a series of papers by Ostrowski (1958-59). These papers considered some properties of the Rayleigh quotient.

As oringinally defined, we assume that X is a real symmetric matrix, U is a row vector, and U^T is its transpose. Then $R(U) = (UXU^T)/(UU^T)$ is called the Rayleigh quotient corresponding to U.

The eigenvalues of X can be characterized by the extremal properties of this quotient. If c_1 is the largest eigenvalue of X, then $c_1 = \max R(U)$, where this maximum is taken over all possible vectors U. If U_1 is the eigenvector corresponding to c_1, then the next largest eigenvalue $c_2 = \max R(U)$, taken over all vectors U that are orthogonal to U_1. The other eigenvalues are similarly characterized.

In practice, what is more important than the extremal properties is the fact that the quotient is stationary. That is, $c_1 = R(U_1)$ differs from $R(U_1 + \epsilon W)$ only by terms of the second order in ϵ. So if we have a vector that is close to the correct eigenvector, the Rayleigh quotient will be quite close to the corresponding eigenvalue.

This makes possible an iterative procedure for determining any specific eigenvalue c. We start with an approximation c_0 that is reasonably close to c. Then we choose an arbitrary vector U_0. The only requirement is that U_0 cannot be orthogonal to the eigenvector U. Then the equations

(5.1) $(X - c_k I)\, U^T_{k+1} = U^T_k$

(5.2) $c_{k+1} = (U_{k+1} X U^T_{k+1})/(U_{k+1} U^T_{k+1})$

are used to get better approximations for c and U. Ostrowski shows that the convergence is cubic, that is, $(c_{k+1} - c)/(c_k - c)^3$ approaches a constant.

If X is not symmetric, we instead consider the generalized Rayleigh quotient $R(U,V) = (VXU^T)/(VU^T)$. In this case, if c is an eigenvalue, U is the corresponding right hand eigenvector $(XU^T = cU^T)$, and V is the corresponding left hand eigenvector $(VX = cV)$, then $R(U,V) = c$. The extremal properties of the Rayleigh quotient no longer hold. However, Ostrowski shows that the quotient is still stationary, if the matrix X has only linear elementary divisors. This will be true if the eigenvalues are distinct, which is the case in fluid mechanics. Moreover, the convergence is still cubic.

The iterative process is now defined by the equations

(5.3) $(X - c_k I)\, U^T_{k+1} = U^T_k$

$$V_{k+1}(X - c_k I) = V_k$$

$$(5.4) \quad c_{k+1} = (V_{k+1} X U^T{}_{k+1})/(V_{k+1} U^T{}_{k+1})$$

Our problem is of the more general form $XU - cYU = 0$. The method of Gary and Helgason could be used to reduce this to the standard form $Y^{-1}XU - cU = 0$. However, the iteration can still be defined if the equation is left in its oringinal form. The equations become

$$(5.5) \quad (X - c_k Y)U^T{}_{k+1} = U^T{}_k$$

$$V_{k+1}(X - c_k Y) = V_k$$

$$(5.6) \quad c_{k+1} = (V_{k+1} X U^T{}_{k+1})/(V_{k+1} Y U^T{}_{k+1})$$

To actually carry out the procedure, we need a starting value c_0. This can be obtained by using the LR algorithm for a smaller value of N, by using an eigenvalue computed for nearby values of α and R, or from the asymtotic analysis. In practice, the asymtotic results are close enough for the procedure to converge.

The matrix $X - c_0 Y$ is then decomposed into the product of a lower triangular matrix times an upper triangular matrix. The same decomposition can be used in obtaining both U and V, since the equation (5.5) becomes

$$(5.7) \quad (\text{Lower})(\text{Upper})U^T{}_{k+1} = U^T{}_k$$

$$(\text{Upper})^T(\text{Lower})^T V^T{}_{k+1} = V^T{}_k$$

where $(\text{Upper})^T$ is lower triangular and $(\text{Lower})^T$ is upper triangular. Since most of the computer time in solving a system of equations is required for the decomposition, this procedure results in a noticeable time savings.

To help insure that U_0 and V_0 are of the proper form, they are defined implicitly by making $(\text{Lower})^{-1}U^T{}_0$ and $((\text{Upper})^T)^{-1}V^T{}_0$ both equal to the vector whose components are all ones. The vectors then depend on the matrix, and the values computed for U_1 and V_1 are unlikely to be orthogonal to U and V.

For the first several iterations, only the equations (5.5) are used. This allows the eigenvectors to converge relatively closely to the proper values. Again, this involves only the oringinal lower upper decomposition, so the procedure is efficient. Then the full iteration can be used until the eigenvalue converges. Generally, the eigenvalue will be accurate to four decimal places after only three or four additional iterations.

The two methods were compared, and they do lead to the same numerical answers. However, the generalized Rayleigh quotient is much more efficient, particularly for large N. For example, if N = 70, the generalized Rayleigh quotient was more than five times as fast as the LR algorithm.

6. NUMERICAL RESULTS

All existing evidence tends to show that plane Couette flow is stable under infinitesimal disturbances. However, because the eigenvalue c is a function of two parameters, the wave number α and the Reynolds number R, it is not easy to systematically cover the entire α, R plane. Moreover, as the product of α and R becomes larger, the Orr-Sommerfeld equation becomes more and more singular, and hence harder to solve numerically. And any instability would be expected to occur for large values of R.

The most recent results on plane Couette flow are due to Davey (1973), who used a combination of asymptotic and numerical methods. Based on the analysis of Wasow and Grohne, Davey studied asymptotically the most familiar case, where $\sqrt{R}$ is much larger than α, and $\sqrt[3]{\alpha R}$ is much larger than 1. Using Airy functions, he arrives at an estimate for the eigenvalue $c = c_r + ic_i$ having the largest imaginary part, given by

$$(6.1) \quad c_r = 1 - 4.1287/(\alpha R)^{1/3}$$
$$c_i = -\alpha/R - 1.0625/(\alpha R)^{1/3}$$

He estimates the error as being of order $R^{-2/3}$.

This indicates that for a given fixed α, as R

approaches infinity, the real part of the eigenvalue approaches one and the imaginary part approaches zero. However, the imaginary part is always negative, indicating that the flow is stable.

Table 1 shows the relative error in the approximation for $\alpha = 1$ and different values of R. As R gets larger, the asymptotic results get closer and closer to the actual values.

TABLE 1

COMPARISON OF ASYMPTOTIC AND COMPUTED VALUES FOR $\alpha = 1$

R	Asymptotic Results	Computed Results	Error
500	.4800 - .1358i	.5092 - .1545i	12.24%
1000	.5871 - .1073i	.6053 - .1192i	3.52%
5000	.7586 - .0623i	.7646 - .0666i	.96%
10000	.8084 - .0492i	.8122 - .0521i	.59%
50000	.8879 - .0288i	.8892 - .0298i	.18%
100000	.9110 - .0229i	.9119 - .0236i	.12%
200000	.9294 - .0182i	.9299 - .0186i	.07%

For the larger values of R, the Orr-Sommerfeld equation is more difficult to solve. But using Chebyshev polynomials, this simply means that N, the degree of the largest Chebyshev polynomial used, must be increased. Thus, if R = 500, N = 40 gives sufficient accuracy, while,

if R = 200000, we must choose N = 80. If larger values of R or more than four place accuracy is required, the value of N then must be chosen larger. A larger value of N does require more computer time, but the process is still quite efficient.

The asymptotic results can be improved by noting that the answers given are always numerically smaller than the actual answers. Using Davey's error estimate, we can create a new approximation

$$(6.2)\quad \begin{aligned} c_r &= 1 - 4.1287/(\alpha R)^{1/3} + R^{-2/3} \\ c_i &= -\alpha/R - 1.0625/(\alpha R)^{1/3} - R^{-2/3} \end{aligned}$$

As R approaches infinity, $R^{-2/3}$ approaches zero, so this doesn't effect the asymptotic theory. However, it does give substantially better numerical results for smaller values of R, as can be seen in Table 2, where the relative error is recomputed using this new approximation. Using these modified results, the generalized Rayleigh quotient normally converges rapidly to an answer.

Davey also studied the less familiar case when $\sqrt[3]{\alpha R}$ is large and α is much larger then $\sqrt{R}$. Again using Airy functions, he arrives at an estimate

$$(6.3)\quad \begin{aligned} c_r &= 1 - 2.0249/(\alpha R)^{1/3} \\ c_i &= -\alpha/R - 1.1691/(\alpha R)^{1/3} \end{aligned}$$

where the error is of order $\alpha^{-5/3}$. This indicates that for a given fixed R, as α approaches infinity, the flow remains stable.

TABLE 2

COMPARISON OF THE MODIFIED ASYMPTOTIC AND COMPUTED VALUES FOR $\alpha = 1$

R	Asymptotic Results	Computed Results	Error
500	.4959 - .1517i	.5092 - .1545i	4.80%
1000	.5971 - .1173i	.6053 - .1192i	1.36%
5000	.7620 - .0657i	.7646 - .0666i	.36%
10000	.8105 - .0516i	.8122 - .0521i	.22%
50000	.8886 - .0295i	.8892 - .0298i	.08%
100000	.9115 - .0234i	.9119 - .0236i	.05%
200000	.9296 - .0184i	.9299 - .0186i	.04%

The asymptotic theory is only valid when either α or R is large. However, if both parameters are small, the Orr-Sommerfeld equation is relatively easy to solve. Gallagher and Mercer (1961) examined numerically the eigenvalues for $\alpha R \leq 1000$. For these small values, the flow is stable.

The other case that is not covered by the asymptotic theory is when α and $\sqrt{R}$ are of the same order of magnitude. This means that the terms of the Orr-Sommerfeld equation are of the same order of magnitude, and no asymptotic simplification is possible.

To explore these values, Davey introduces new variables

$$(6.4)\quad X = \alpha R^{-1/2}$$

$$Y = -R^{1/2} c_1$$

Then the second part of (6.1) becomes

$$(6.5)\quad Y = X + 1.0625X^{-1/3} + O(X^{1/3})$$

Where X is small (that is, α is much smaller than $\sqrt{R}$) and R is large. The second part of (6.3) becomes

$$(6.6)\quad Y = X + 1.1691X^{-1/3} + O(X^{-5/3})$$

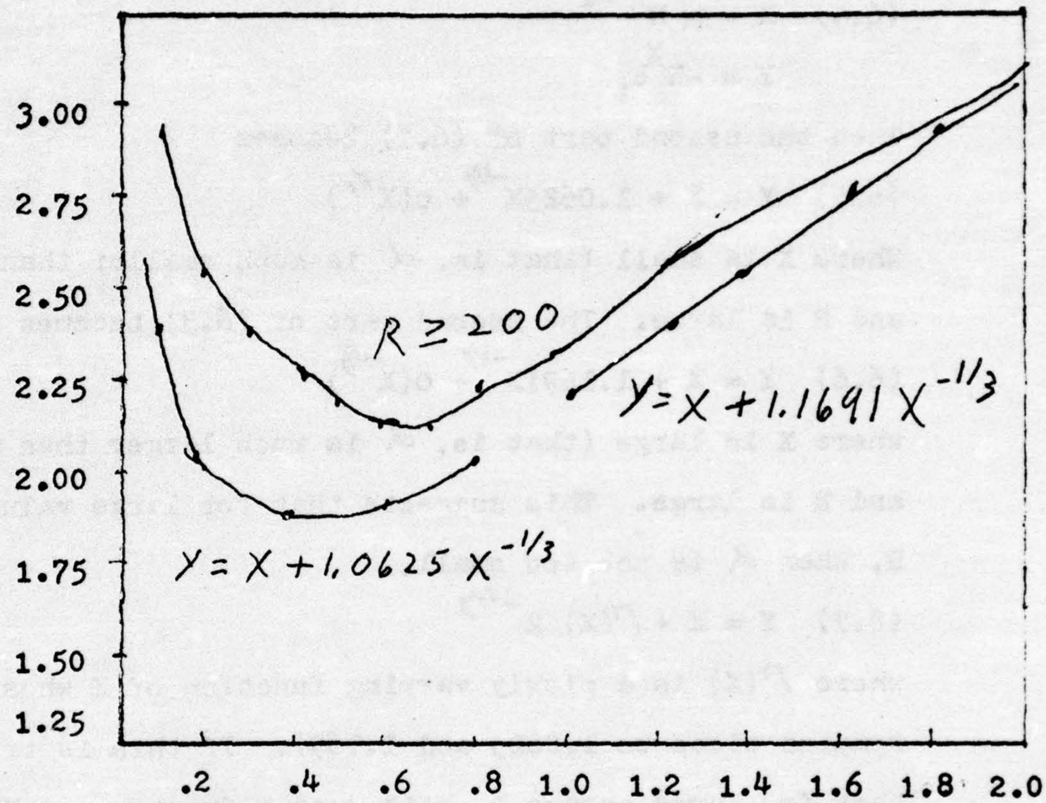
where X is large (that is, α is much larger than $\sqrt{R}$) and R is large. This suggests that for large values of R, when α is not too small,

$$(6.7)\quad Y = X + \Gamma(X)\, X^{-1/3}$$

where $\Gamma(X)$ is a slowly varying function of X whose value remains close to 1.0625 and 1.1691. If this is true, then for large enough R, Y is just a function of X. If Y is always positive, c_1 will be negative, and the flow will be stable to infinitesimal disturbances for all values of α and R.

Davey claims that this is the case for $R \geq 200$. While not giving the exact numerical data, he states he has computed the eigenvalues for values of αR up to 100000, and the values for Y depended solely on X. The resulting curve is given in Figure 1, which also shows the curves predicted by the asymptotic equations (6.5) and (6.7) for comparison.

Fig. 1.--Graph of Y versus X



To check this result, eigenvalues were computed for values of αR up to 250000. The results for R = 200, R = 1000, R = 3000, and R = 5000 are given in the following tables. The results show a high degree of consistency. For the larger values of αR the results do not match exactly. However, in this case a small error in c_1 can lead to a fairly large error in Y. And the general principle that Y depends on X seems to be well established. Any variance in the values of Y appears to be quite small compared to the size of Y.

TABLE 3

VALUES OF X, α, c, AND Y FOR R = 200 AND N = 50

X	α	eigenvalue	Y
.1	1.41	.4297 - .2032i	2.874
.2	2.83	.5748 - .1789i	2.530
.3	4.24	.6480 - .1667i	2.357
.4	5.66	.6932 - .1594i	2.254
.5	7.07	.7242 - .1555i	2.199
.6	8.49	.7471 - .1540i	2.178
.7	9.90	.7649 - .1543i	2.182

TABLE 4

VALUES OF X, α, c, AND Y FOR R = 1000 and N = 60

X	α	eigenvalue	Y
.1	3.16	.7444 - .0909i	2.874
.2	6.32	.8098 - .0800i	2.530
.3	9.49	.8425 - .0745i	2.356
.4	12.65	.8627 - .0713i	2.254
.5	15.81	.8766 - .0695i	2.198
.6	18.97	.8869 - .0689i	2.178
.7	22.14	.8948 - .0690i	2.182

TABLE 5

VALUES OF X, α, c, AND Y FOR R = 3000 AND N = 70

X	α	eigenvalue	Y
.1	5.48	.8524 - .0525i	2.873
.2	10.95	.8902 - .0462i	2.529
.3	16.43	.9091 - .0430i	2.354
.4	21.91	.9208 - .0412i	2.258
.5	27.39	.9288 - .0401i	2.194
.6	32.86	.9345 - .0397i	2.173
.7	38.34	.9390 - .0400i	2.191

TABLE 6

VALUES OF X, α, c, AND Y FOR R = 5000 AND N = 80

X	α	eigenvalue	Y
.1	7.07	.8857 - .0407i	2.875
.2	14.14	.9149 - .0358i	2.529
.3	21.21	.9296 - .0333i	2.356
.4	28.28	.9386 - .0319i	2.259
.5	35.36	.9448 - .0310i	2.189
.6	42.43	.9491 - .0306i	2.167
.7	49.50	.9527 - .0310i	2.189

To check this, some additional values were run for $N = 90$. For this case, if $R = 3000$ and $X = .4$, then $Y = 2.254$, which exactly matches the value for $R = 200$ and $X = .4$. If $R = 3000$ and $X = .7$, then $Y = 2.181$, which is much closer to the proper value. And if $R = 5000$ and $X = .4$, then $Y = 2.258$, which is slightly closer to the expected value. So if N is chosen large enough, the values of Y do appear to agree.

For a given fixed value of R, the minimum value of $-c_1$ occurs between $X = .6$ and $X = .7$. As R gets larger, c_1 will approach zero, but it will always be negative. The belief that Couette flow is stable under infinitesimal disturbances is further confirmed.

7. THE NON-LINEAR ANALYSIS

The linear case predicts stability for all values of α and R. However, as has been mentioned, Reichardt (1956) was able to maintain laminar flow only for values of R up to about 750.

This could be due to various causes. First, the apparatus could only approximate an infinite channel. Also, the flow in an experiment is three dimensional, and vortices perpendicular to the direction of the flow could affect the results. Finally, the perturbations are of course finite, and there is no exact way of measuring their amplitude or wave number. In this section of the paper I will try to take into account this last effect, by assuming a non-linear solution.

The main approach to this type of problem was developed by J. T. Stuart and J. Watson in a series of papers in the early 1960's. We will use here in general the notation of Reynolds and Potter (1967), who modified somewhat the approach of Stuart and Watson.

We will again consider the Navier-Stokes equations

(7.1) $u_t + uu_x + vu_y + p_x - (u_{xx} + u_{yy})/R = 0$

(7.2) $v_t + uv_x + vv_y + p_y - (v_{xx} + v_{yy})/R = 0$

(7.3) $u_x + v_y = 0$

with the boundary conditions

(7.4) $u(x, \pm 1, t) = \pm 1$

$v(x, \pm 1, t) = 0$

Then we attempt to solve this in terms of the basic linear stream function $\psi(x,y,t) = \phi(y) \exp(i\alpha(x - ct))$ and its harmonics, under the assumption that the linear solution is a reasonable approximation to the actual solution.

In the general case, the amplitude of the disturbance becomes important, so we will try to isolate this term. The linear solution can be rewritten as

(7.5) $\psi(x,y,t) = \phi(y) \exp(i\alpha(x - c_r t)) \exp(\alpha c_i t)$

In this case, the magnitude of the disturbance is determined by $\phi(y)$ and $\exp(\alpha c_i t)$, which are independent. The Orr-Sommerfeld equation determines $\phi(y)$ only up to an arbitrary multiple, so it can be normalized in any convenient fashion. This determines the relative size of the disturbance for different values of y, but the absolute size of the disturbance is immaterial. The term $\exp(\alpha c_i t)$ determines how the disturbance behaves with respect to time, that is, whether it will die out or increase to the stage where the linear theory no longer applies. As far as we are concerned, this is the important part of the solution.

To try to isolate a similar function in the general case, we introduce a change of variables

(7.6) $\theta = \alpha x + \omega t$

$A = A(t)$

A $\omega = \omega(t)$

Here θ represents the periodic part of the solution, α as usual represents the wavenumber, and ω represents the frequency of the solution, In the linear case, $\theta = \alpha x - \alpha c_r t$ and $\omega = -\alpha c_r$. In the non-linear case, the frequency ω will depend on the size of the disturbance. So ω is viewed as a function of a yet to be defined amplitude function A, which depends only on the time t. By analogy, the function A(t) will show how the disturbance changes for any fixed y with respect to time. In the linear case, $A(t) = \exp(\alpha c_i t)$.

With this change of variables, the Navier-Stokes equations become, with u and v considered as functions of y, θ, A

$$(7.7)\quad \frac{dA}{dt}\frac{\partial u}{\partial A} + \left[\omega + t\frac{\partial \omega}{\partial A}\frac{dA}{dt} + \alpha u\right]\frac{\partial u}{\partial \theta} + v\frac{\partial u}{\partial y} + \alpha\frac{\partial p}{\partial \theta} - \frac{1}{R}\left[\alpha^2\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0$$

$$(7.8)\quad \frac{dA}{dt}\frac{\partial v}{\partial A} + \left[\omega + t\frac{\partial \omega}{\partial A}\frac{dA}{dt} + \alpha u\right]\frac{\partial v}{\partial \theta} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} - \frac{1}{R}\left[\alpha^2\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial y^2}\right] = 0$$

$$(7.9)\quad \alpha\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial y} = 0$$

As before, we eliminate the pressure term by differentiating (7.7) with respect to y and multiplying (7.8) by α and differentiating it with respect to θ, and subtracting the equations. This results in

$$\text{(7.10)} \quad \frac{dA}{dt}\frac{\partial^2 u}{\partial A \partial y} + \left[w + t\frac{\partial w}{\partial A}\frac{dA}{dt} + \alpha u\right]\frac{\partial^2 u}{\partial y \partial \theta}$$

$$+ \alpha \frac{\partial u}{\partial y}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^2 p}{\partial \theta \partial y}$$

$$-\frac{1}{R}\left[\alpha^2\frac{\partial^3 u}{\partial \theta^2 \partial y} + \frac{\partial^3 u}{\partial y^3}\right] - \alpha\frac{dA}{dt}\frac{\partial^2 v}{\partial A \partial \theta}$$

$$-\alpha\left[w + t\frac{\partial w}{\partial A}\frac{dA}{dt} + \alpha u\right]\frac{\partial^2 v}{\partial \theta^2} - \alpha\frac{\partial u}{\partial \theta}\frac{\partial v}{\partial \theta} - \alpha\frac{\partial v}{\partial \theta}\frac{\partial v}{\partial y}$$

$$-\alpha v\frac{\partial^2 v}{\partial y \partial \theta} - \alpha\frac{\partial^2 p}{\partial \theta \partial y} + \frac{1}{R}\left[\alpha^3\frac{\partial^3 v}{\partial \theta^3} + \alpha\frac{\partial^3 v}{\partial y^2 \partial \theta}\right] = 0$$

Again we introduce a stream function $\psi(A, y, \theta)$ defined by

$$\text{(7.11)} \quad \frac{\partial \psi}{\partial y} = \alpha u \qquad \frac{\partial \psi}{\partial \theta} = -v$$

Also, for convenience we introduce the notation

$$\text{(7.12)} \quad \varphi(A, y, \theta) = \frac{\partial^2 \psi}{\partial y^2} + \alpha^2\frac{\partial^2 \psi}{\partial \theta^2}$$

Substituting and multiplying by α, equation (7.10) becomes

$$\text{(7.13)} \quad \frac{dA}{dt}\frac{\partial \varphi}{\partial A} + \left[w + t\frac{dw}{dA}\frac{dA}{dt} + \frac{\partial \psi}{\partial y}\right]\frac{\partial \varphi}{\partial \theta}$$

$$-\frac{\partial \psi}{\partial \theta}\frac{\partial \varphi}{\partial y} - \frac{1}{R}\left[\frac{\partial^2 \varphi}{\partial y^2} + \alpha^2\frac{\partial^2 \varphi}{\partial \theta^2}\right] = 0$$

The continuity equation (7.3) becomes

$$\text{(7.14)} \quad \frac{\partial \psi^2}{\partial \theta \partial y} - \frac{\partial^2 \psi}{\partial \theta \partial y} = 0$$

so it is automatically satisfied.

The boundary conditions (7.4) become

$$\text{(7.15)} \quad \frac{\partial \psi}{\partial \theta}(A, \pm 1, \theta) = 0$$

$$\frac{\partial \psi}{\partial y}(A, \pm 1, \theta) = \pm\alpha$$

We can now expand the stream function as a Fourier series

$$(7.16)\quad \psi(A,y,\theta) = \sum_{k=0}^{\infty}\left[\psi^{(k)}(A,y)e^{ik\theta} + \tilde{\psi}^{(k)}(A,y)e^{-ik\theta}\right]$$

where $\tilde{\psi}^{(k)}$ is the complex conjugate of $\psi^{(k)}$. So while the $\psi^{(k)}(A,y)$ are complex valued, $\psi(A,y,\theta)$ is real valued. To reduce somewhat the mathematical complexity of the problem, we will terminate the expansion at k = 3. Since the linear case has been chosen as satisfying the Orr-Sommerfeld equation, hopefully this will already be reasonably accurate. Note that $\psi^{(0)}(A,y)$ can be chosen real valued, since any imaginary part will cancel out.

So we have.

$$(7.17)\quad \psi(A,y,\theta) = 2\psi^{(0)} + \psi^{(1)}e^{i\theta} + \tilde{\psi}^{(1)}e^{-i\theta} + \psi^{(2)}e^{2i\theta} + \tilde{\psi}^{(2)}e^{-2i\theta} + \psi^{(3)}e^{3i\theta} + \tilde{\psi}^{(3)}e^{-3i\theta}$$

We introduce the notation

$$(7.18)\quad Z^{(k)}(A,y) = \left[\frac{\partial^2}{\partial y^2} - \alpha^2 k^2\right]\psi^{(k)}(A,y)$$

This allows us to write

$$(7.19)\quad \zeta(A,y,\theta) = 2Z^{(0)} + Z^{(1)}e^{i\theta} + \tilde{Z}^{(1)}e^{-i\theta} + Z^{(2)}e^{2i\theta} + \tilde{Z}^{(2)}e^{-2i\theta} + Z^{(3)}e^{3i\theta} + \tilde{Z}^{(3)}e^{-3i\theta}$$

We substitute (7.17) and (7.19) into (7.13) and equate the coefficients of the constant, $e^{i\theta}$, $e^{2i\theta}$, and $e^{3i\theta}$ terms. We will ignore all terms of order greater than three.

The last of these four equations will not be required in our procedure, so we will just give the first three equations

(7.20) $$2\frac{\partial A}{\partial t}\frac{\partial Z^{(0)}}{\partial A} - i\frac{\partial \psi^{(1)}}{\partial y}\tilde{Z}^{(1)} + i\frac{\partial \tilde{\psi}^{(1)}}{\partial y}Z^{(1)} - i\psi^{(1)}\frac{\partial \tilde{Z}^{(1)}}{\partial y} + i\tilde{\psi}^{(1)}\frac{\partial Z^{(1)}}{\partial y} - \frac{2}{R}\frac{\partial^2 Z^{(0)}}{\partial y^2} = 0$$

(7.21) $$\frac{dA}{dt}\frac{\partial Z^{(1)}}{\partial A} + i\left[\omega + t\frac{d\omega}{dA}\frac{dA}{dt}\right]Z^{(1)} + 2i\frac{\partial \psi^{(0)}}{\partial y}Z^{(1)} + 2i\frac{\partial \tilde{\psi}^{(1)}}{\partial y}Z^{(2)} - i\frac{\partial \psi^{(2)}}{\partial y}\tilde{Z}^{(1)} - 2i\psi^{(1)}\frac{\partial Z^{(0)}}{\partial y} + i\tilde{\psi}^{(1)}\frac{\partial Z^{(2)}}{\partial y} - 2i\psi^{(2)}\frac{\partial \tilde{Z}^{(1)}}{\partial y} - \frac{1}{R}\left[\frac{\partial^2 Z^{(1)}}{\partial y^2} - \alpha^2 Z^{(1)}\right] = 0$$

(7.22) $$\frac{dA}{dt}\frac{\partial Z^{(2)}}{\partial A} + \left[\omega + t\frac{d\omega}{dA}\frac{dA}{dt}\right]2iZ^{(2)} + 4i\frac{\partial \psi^{(0)}}{\partial y}Z^{(2)} + i\frac{\partial \psi^{(1)}}{\partial y}Z^{(1)} - i\psi^{(1)}\frac{\partial Z^{(1)}}{\partial y} - 4i\psi^{(2)}\frac{\partial Z^{(0)}}{\partial y} - \frac{1}{R}\left[\frac{\partial^2 Z^{(2)}}{\partial y^2} - 4\alpha^2 Z^{(2)}\right] = 0$$

The boundary conditions (7.15) become

(7.23) $$2\frac{\partial \psi^{(0)}}{\partial y}(A, \pm 1) = \pm\alpha$$
$$\psi^{(k)}(A, \pm 1) = \frac{\partial \psi^{(k)}}{\partial y}(A, \pm 1) = 0 \qquad k = 1, 2, 3$$

The fact that the equations are non-linear and coupled makes the solution difficult. However, assuming that the amplitude is small, we can seek a solution as a power series in A. We require that the solution for infinitesimal amplitude reduce to the Orr-Sommerfeld wave, and the solution for zero amplitude to the basic laminar solution. This suggests seeking a solution in the form

(7.24) $$\psi^{(k)}(A,y) = \sum_{n=k}^{\infty} A^n \phi^{(k;n)}(y)$$

We will show that this satisfies the above conditions. Again, we will truncate the expansion at n = 3, and k will only have the values 0,1,2, and 3.

We also expand as a power series

(7.25) $$\frac{dA}{dt} = a^{(0)}A + a^{(1)}A^2 + a^{(2)}A^3 + \cdots$$

For infinitesimal A, the amplitude behaves exponentially, as in the linear theory.

Finally we represent

(7.26) $$\omega + t\,\frac{d\omega}{dA}\,\frac{dA}{dt} = b^{(0)} + b^{(1)}A + b^{(2)}A^2 + \cdots$$

To simplify the equations somewhat, we introduce the notation

(7.27) $$Z^{(k;n)} = (D^2 - \alpha^2 k^2)\,\phi^{(k;n)}$$

This allows us to write

(7.28) $$Z^{(k)} = \sum_{n=k}^{3} A^n Z^{(k;n)}$$

We substitute these expansions into the equations (7.20), (7.21), and (7.22), and equate the coefficients of the powers of A. We will go only up to the A^3 terms. Equation (7.20) becomes the four equations

(7.29) $$-\frac{2}{R}\,D^2 Z^{(0;0)} = 0$$

(7.30) $$2a^{(0)}Z^{(0;1)} - \frac{2}{R}D^2 Z^{(0;1)} = 0$$

(7.31) $$4a^{(0)}Z^{(0;2)} + 2a^{(1)}Z^{(0;1)} - iD\phi^{(1;1)}\tilde{Z}^{(1;1)}$$
$$+ iD\tilde{\phi}^{(1;1)}Z^{(1;1)} - i\phi^{(1;1)}D\tilde{Z}^{(1;1)} + i\tilde{\phi}^{(1;1)}DZ^{(1;1)} - \frac{2}{R}D^2 Z^{(0;2)} = 0$$

(7.32) $$6a^{(0)}Z^{(0;3)} + 4a^{(1)}Z^{(0;2)} + 2a^{(2)}Z^{(0;1)} - iD\phi^{(1;1)}\tilde{Z}^{(1;2)}$$
$$- iD\phi^{(1;2)}\tilde{Z}^{(1;1)} + iD\tilde{\phi}^{(1;1)}Z^{(1;2)} + iD\tilde{\phi}^{(1;2)}Z^{(1;1)} - i\phi^{(1;1)}D\tilde{Z}^{(1;2)}$$
$$- i\phi^{(1;2)}D\tilde{Z}^{(1;1)} + i\tilde{\phi}^{(1;2)}DZ^{(1;1)} + i\tilde{\phi}^{(1;1)}DZ^{(1;2)} - \frac{2}{R}D^2 Z^{(0;3)} = 0$$

Equation (7.21) becomes the three equations

(7.33) $$a^{(0)}Z^{(1;1)} + ib^{(0)}Z^{(1;1)} + 2iD\phi^{(0;0)}Z^{(1;1)}$$
$$- 2i\phi^{(1;1)}DZ^{(0;0)} - \frac{1}{R}\left[D^2 - \alpha^2\right]Z^{(1;1)} = 0$$

(7.34) $$2a^{(0)}Z^{(1;2)} + a^{(1)}Z^{(1;1)} + ib^{(0)}Z^{(1;2)}$$
$$+ ib^{(1)}Z^{(1;1)} + 2iD\phi^{(0;0)}Z^{(1;2)} + 2iD\phi^{(0;1)}Z^{(1;1)}$$
$$- 2i\phi^{(1;1)}DZ^{(0;1)} - 2i\phi^{(1;2)}DZ^{(0;0)} - \frac{1}{R}\left[D^2 - \alpha^2\right]Z^{(1;2)} = 0$$

(7.35) $$3a^{(0)}Z^{(1;3)} + 2a^{(1)}Z^{(1;2)} + a^{(2)}Z^{(1;1)}$$
$$+ ib^{(0)}Z^{(1;3)} + ib^{(1)}Z^{(1;2)} + ib^{(2)}Z^{(1;1)}$$
$$+ 2iD\phi^{(0;0)}Z^{(1;3)} + 2iD\phi^{(0;1)}Z^{(1;2)} + 2iD\phi^{(0;2)}Z^{(1;1)}$$
$$+ 2iD\tilde{\phi}^{(1;1)}Z^{(2;2)} - iD\phi^{(2;2)}\tilde{Z}^{(1;1)} - 2i\phi^{(1;1)}DZ^{(0;2)} - 2i\phi^{(1;2)}DZ^{(0;1)}$$
$$- 2i\phi^{(1;3)}DZ^{(0;0)} + i\tilde{\phi}^{(1;1)}DZ^{(2;2)} - 2i\phi^{(2;2)}D\tilde{Z}^{(1;1)} - \frac{1}{R}\left[D^2 - \alpha^2\right]Z^{(1;3)} = 0$$

Equation (7.22) becomes two equations, but only the first

$$(7.36)\quad 2a^{(0)}Z^{(2;2)} + 2ib^{(0)}Z^{(2;2)} + 4iD\phi^{(0;0)}Z^{(2;2)}$$
$$+ iD\phi^{(1;1)}Z^{(1;1)} - i\phi^{(1;1)}DZ^{(1;1)} - 4i\phi^{(2;2)}DZ^{(0;0)} - \frac{1}{R}\left[D^2 - 4\alpha^2\right]Z^{(2;2)} = 0$$

will be needed.

The boundary conditions (7.23) become

$$(7.37)\quad \phi^{(k;n)}(\pm 1) = D\phi^{(k;n)}(\pm 1) = 0 \qquad k = 1,2,3 \quad n = k \ldots 3$$
$$D\phi^{(0;0)}(\pm 1) = \pm\frac{\alpha}{2}$$
$$D\phi^{(0;n)}(\pm 1) = 0 \qquad n = 1,2,3$$

We do not have boundary conditions for $\phi^{(0;0)}$, $\phi^{(0;1)}$, $\phi^{(0;2)}$ and $\phi^{(0;3)}$. This depends on the normalization. What will be most convenient is to set the bulk average velocity equal to zero, that is, $\int_{-1}^{1} u(x,y,t)dy = 0$. This is true for the laminar case. Then

$$\frac{1}{\alpha}\int_{-1}^{1}\frac{\partial\psi}{\partial y}dy = 0$$
$$\psi(A,1,\theta) - \psi(A,-1,\theta) = 0$$
$$\psi^{(0)}(A,1) - \psi^{(0)}(A,-1) = 0$$

This can most easily be satisfied by choosing

$$(7.38)\quad \phi^{(0;n)}(\pm 1) = 0 \qquad n = 0,1,2,3$$

We will now try to solve out set of equations. Equation (7.29) with its boundary conditions can be written as

$$(7.39)\quad D^4 \phi^{(0;0)}(y) = 0$$

$$\phi^{(0;0)}(\pm 1) = 0 \qquad D\phi^{(0;0)}(\pm 1) = \pm \frac{\alpha}{2}$$

This has the solution

$$(7.40)\quad \phi^{(0;0)} = \frac{\alpha}{4} y^2 - \frac{\alpha}{4}$$

Note that if the amplitude of the disturbance A = 0, we get $\psi(A, y, \theta) = 2\phi^{(0;0)}(y) = \alpha y^2/2 - \alpha/2$ and $u(x,y,t) = y$ and $v(x,y,t) = 0$. This is the basic laminar solution, as desired.

Equation (7.30) becomes

$$(7.41)\quad 2a^{(0)} D^2 \phi^{(0;1)} - \frac{2}{R} D^4 \phi^{(0;1)} = 0$$

$$\phi^{(0;0)}(\pm 1) = D\phi^{(0;1)}(\pm 1) = 0$$

So the solution is

$$(7.42)\quad \phi^{(0;1)}(y) = 0$$

Equation (7.33) becomes

$$(7.43)\quad \left\{ (D^2 - \alpha^2)^2 - i\alpha R \left(y - \frac{-b^{(0)} + ia^{(0)}}{\alpha} \right) (D^2 - \alpha^2) \right\} \phi^{(1;1)} = 0$$

$$\phi^{(1;1)}(\pm 1) = D\phi^{(1;1)}(\pm 1) = 0$$

This is the Orr-Sommerfeld equation with $c = (-b^{(0)} + ia^{(0)})/\alpha$, that is

$$(7.44)\quad b^{(0)} = -\alpha c_r$$

$$a^{(0)} = \alpha c_i$$

If we assume an infinitesimal disturbance, and ignore all but the terms that are linear in A, then $dA/dt = a^{(0)} A = \alpha c_1 A$ or $A = \exp(\alpha c_1 t)$. Also, $\psi^{(0)} = \alpha y^2/4 - \alpha/4$ and $\psi^{(1)} = \phi^{(1;1)} \exp(\alpha c_1 t)$. $b^{(1)}$ will later be shown to be zero, so if we assume that ω is a constant (that is, the frequency of the wave doesn't change), then $\omega = b^{(0)} = -\alpha c_r$. This gives us

$$\psi(A, y, \theta) = \frac{\alpha}{2} y^2 - \frac{\alpha}{2} + \phi^{(1;1)} e^{i\alpha(x-ct)} + \tilde{\phi}^{(1;1)} e^{-i\alpha(x+ct)}$$

So our formalism is an extension of what has been done in the linear case.

$\phi^{(1;1)}(y)$ must be normalized in some fashion. We will use the standard normalization

$$(7.45) \qquad \phi^{(1;1)}(0) = 1$$

This will implicitly determine the function A(t), which is the other part of the solution relating to the size of the disturbance.

Continuing our discussion of the separate equations, equation (7.31) becomes

$$(7.46) \quad \frac{2}{R} D^4 \phi^{(0;2)} - 4a^{(0)} D^2 \phi^{(0;2)} = -i D\phi^{(1;1)}(D^2-\alpha^2)\tilde{\phi}^{(1;1)} + iD\tilde{\phi}^{(1;1)}(D^2-\alpha^2)\phi^{(1;1)} - i\phi^{(1;1)} D(D^2-\alpha^2)\tilde{\phi}^{(1;1)} + i\tilde{\phi}^{(1;1)} D(D^2-\alpha^2)\phi^{(1;1)}$$

$$\phi^{(0;2)}(\pm 1) = D\phi^{(0;2)}(\pm 1) = 0$$

This is an inhomogeneous differential equation. If $\phi^{(1;1)}$ is known, this can be solved for $\phi^{(0;2)}$.

Similarly, Equation (7.36) becomes

$$(7.47)\left\{(D^2-4\alpha^2)^2-2i\alpha R\left(y-\frac{-b^{(0)}+ia^{(0)}}{\alpha}\right)(D^2-4\alpha^2)\right\}\phi^{(2;2)}$$

$$=iRD\phi^{(1;1)}(D^2-\alpha^2)\phi^{(1;1)}-iR\phi^{(1;1)}D(D^2-\alpha^2)\phi^{(1;1)}$$

$$\phi^{(2;2)}(\pm 1)=D\phi^{(2;2)}(\pm 1)=0$$

This is also an inhomogeneous differential equation.

The solution of the last three equations (7.32), (7.34), and (7.35) requires some knowledge of the constants $a^{(1)}$, $a^{(2)}$, $b^{(1)}$, and $b^{(2)}$. For example, (7.34) becomes

$$(7.48)\left\{(D^2-\alpha^2)^2-i\alpha R\left(y-\frac{-b^{(0)}+2ia^{(0)}}{\alpha}\right)(D^2-\alpha^2)\right\}\phi^{(1;2)}$$

$$=R\,(a^{(1)}+ib^{(1)})\,(D^2-\alpha^2)\,\phi^{(1;1)}$$

$$\phi^{(1;2)}(\pm 1)=D\phi^{(1;2)}(\pm 1)=0$$

The difficulty is that there is not enough information to specify the values of $a^{(1)}$ and $b^{(1)}$. So we must make some additional assumption to guarantee a unique answer.

This would not be a problem is $a^{(0)}=\alpha\, c_1 = 0$. In this case the operator acting on $\phi^{(1;2)}$ would be the Orr-Sommerfeld operator. So the adjoint homogeneous equation would have a solution, and (7.48) could not be solved unless its right side was orthogonal to this solution. This consistency condition would determine $a^{(1)}$ and $b^{(1)}$.

This can be done in the related case of plane Poiseuille flow, where this method has been previously applied. Here there is a stability curve for the linear

case in the α-R plane, where $c_i = 0$. On this curve, the constants are determined by the orthogonality condition. Near the neutral curve the same method is used based on continuity. In this case $a^{(1)}$ turns out to be zero, and $dA/dt = a^{(2)} A^3 + \ldots$. So the sign of $a^{(2)}$ determines whether or not the flow is stable and the disturbance dies out ($a^{(2)} < 0$) or unstable and the disturbance grows ($a^{(2)} > 0$).

In the case of Couette flow the imaginary part of the eigenvalue is never equal to zero. So we will apply an alternate method, also suggested by Reynolds and Potter.

Instead of insisting that the linear flow be at the transition point, we will insist that the non-linear flow be at the transition point, that is, $dA/dt = 0$. So $A(t)$ will be a constant, and the disturbance will neither grow nor decay. The solution will be steady state with frequency ω.

To solve this problem, we expand

$$\omega = \omega^{(0)} + A\omega^{(1)} + A^2 \omega^{(2)} + \cdots \tag{7.49}$$

While ω is a real number, we will view it as complex as a mathematical convenience. The physical problem will only have solutions for certain values of A.

Our procedure is the same up to the equations (7.20), (7.21), and (7.22). We now set dA/dt equal to zero and substitute in the expansion for ω. This creates minor changes in the set of equations (7.29) thru (7.36).

Equation (7.29) remains the same, so the solution for $\phi^{(0;0)}$ given in (7.40) also remains unchanged.

The first term of equation (7.30) disappears, but the solution for $\phi^{(0;1)}$ given by (7.42) is still the same.

For equation (7.33) the term $a^{(0)}z^{(1;1)}$ drops out, and instead of the form given by (7.43) we obtain the similar form

$$(7.50)\ \left\{(D^2-\alpha^2)^2 - i\alpha R\left(y+\frac{w^{(0)}}{\alpha}\right)(D^2-\alpha^2)\right\}\phi^{(1;1)}=0$$

$$\phi^{(1;1)}(\pm 1) = D\phi^{(1;1)}(\pm 1) = 0$$

This is still the Orr-Sommerfeld equation, but now $w^{(0)} = -\alpha c$, where c is the eigenvalue of the Orr-Sommerfeld equation.

For equation (7.31), the $a^{(0)}$ and $a^{(1)}$ terms drop out, and (7.46) simplifies slightly to become

$$(7.51)\ \frac{2}{R}D^4\phi^{(0;2)} = -iD\phi^{(1;1)}(D^2-\alpha^2)\tilde{\phi}^{(1;1)}$$

$$+iD\tilde{\phi}^{(1;1)}(D^2-\alpha^2)\phi^{(1;1)} - i\phi^{(1;1)}D(D^2-\alpha^2)\tilde{\phi}^{(1;1)} + i\tilde{\phi}^{(1;1)}D(D^2-\alpha^2)\phi^{(1;1)}$$

$$\phi^{(0;2)}(\pm 1) = D\phi^{(0;2)}(\pm 1) = 0$$

Similarly, the $a^{(0)}$ terms drops out of (7.36), and (7.47) becomes

$$(7.52)\ \left\{(D^2-4\alpha^2)^2 - 2i\alpha R\left(y+\frac{w^{(0)}}{\alpha}\right)(D^2-4\alpha^2)\right\}\phi^{(2;2)}$$

$$= iRD\phi^{(1;1)}(D^2-\alpha^2)\phi^{(1;1)} - iR\phi^{(1;1)}D(D^2-\alpha^2)\phi^{(1;1)}$$

$$\phi^{(2;2)}(\pm 1) = D\phi^{(2;2)}(\pm 1) = 0$$

The left side of this equation is the Orr-Sommerfeld operator, but with α replaced by 2α.

Equation (7.34) was the first of the equations that presented difficulties. But now with the $a^{(0)}$ term removed, (7.48) can be rewritten as

$$(7.53)\ \left\{(D^2-\alpha^2)^2 - i\alpha R\left(y + \frac{w^{(0)}}{\alpha}\right)(D^2-\alpha^2)\right\}\phi^{(1;2)}$$
$$= i\,w^{(1)} R\,(D^2-\alpha^2)\,\phi^{(1;1)}$$
$$\phi^{(1;2)}(\pm 1) = D\phi^{(1;2)}(\pm 1) = 0$$

The left side of this equation is just the Orr-Sommerfeld operator acting on $\phi^{(1;2)}$. We know that the corresponding homogeneous equation has a non-trivial solution $\phi^{(1;1)}$. So the adjoint homogeneous system, which has the form

$$(7.54)\ \left\{(D^2-\alpha^2)^2 - i\alpha R\left[\left(y + \frac{w^{(0)}}{\alpha}\right)(D^2-\alpha^2) + 2D\right]\right\}\phi = 0$$
$$\phi(\pm 1) = D\phi(\pm 1) = 0$$

has a non-trivial solution. We will call this solution v(y).

The inhomogeneous system will have a solution only if the consistency condition

$$(7.55)\quad i\,w^{(1)}\int_{-1}^{1}\left[(D^2-\alpha^2)\,\phi^{(1;1)}(y)\right]v(y)\,dy = 0$$

is satisfied. This is unlikely to be true unless $w^{(1)} = 0$. Then $\phi^{(1;2)}$ can be any multiple of $\phi^{(1;1)}$. For convenience

we will choose the solution

(7.56) $$\phi^{(1;2)}(y) = 0$$

This leaves two more out of our oringinal set of eight equations to consider. Equation (7.32) now reduces to

(7.57) $$D^4 \phi^{(0;3)}(y) = 0$$

$$\phi^{(0;3)}(\pm 1) = D\phi^{(0;3)}(\pm 1) = 0$$

so the solution is simply

(7.58) $$\phi^{(0;3)}(y) = 0$$

The last equation (7.35) can be written

(7.59) $$\left\{(D^2-\alpha^2)^2 - i\alpha R\left(y + \frac{w^{(0)}}{\alpha}\right)(D^2-\alpha^2)\right\}\phi^{(1;3)}$$

$$= iR\{w^{(2)}(D^2-\alpha^2)\phi^{(1;1)} + 2D\phi^{(0;2)}(D^2-\alpha^2)\phi^{(1;1)}$$

$$+ 2D\tilde{\phi}^{(1;1)}(D^2-4\alpha^2)\phi^{(2;2)} - D\phi^{(2;2)}(D^2-\alpha^2)\tilde{\phi}^{(1;1)} - 2\phi^{(1;1)}D^3\phi^{(0;2)}$$

$$+ \tilde{\phi}^{(1;1)}D(D^2-4\alpha^2)\phi^{(2;2)} - 2\phi^{(2;2)}D(D^2-\alpha^2)\tilde{\phi}^{(1;1)}$$

$$\phi^{(1;3)}(\pm 1) = D\phi^{(1;3)}(\pm 1) = 0$$

The left side is still the Orr-Sommerfeld operator, so we have a consistency condition

(7.60) $$w^{(2)}\int_{-1}^{1}[(D^2-\alpha^2)\phi^{(1;1)}(y)]\,v(y)\,dy =$$

$$\int_{-1}^{1} v(y)[-2D\phi^{(0;2)}(D^2-\alpha^2)\phi^{(1;1)} - 2D\tilde{\phi}^{(1;1)}(D^2-4\alpha^2)\phi^{(2;2)}$$

$$+D\phi^{(2;2)}(D^2-\alpha^2)\tilde{\phi}^{(1;1)}+2\phi^{(1;1)}D^3\phi^{(0;2)}$$
$$-\tilde{\phi}^{(1;1)}D(D^2-4\alpha^2)\phi^{(2;2)}+2\phi^{(2;2)}D(D^2-\alpha^2)\tilde{\phi}^{(1;1)}]\,dy$$

This determines the value for $w^{(2)}$, which is what we are actually after. Additional values can be gotten using the same procedures, but $w^{(3)}$ will turn out to be zero, and to obtain $w^{(4)}$ would require much more numerical work.

So we have the relation

$$w = w^{(0)} + w^{(2)}A^2 \tag{7.61}$$

Since w and A are both real, this means

$$w = \mathrm{Real}(w^{(0)}) + \mathrm{Real}(w^{(2)})A^2 \tag{7.62}$$
$$0 = \mathrm{Imag}(w^{(0)}) + \mathrm{Imag}(w^{(2)})A^2$$

This determines the size of the amplitude A, which is given by

$$A = \sqrt{\frac{\alpha c_i}{\mathrm{Imag}(w^{(2)})}} \tag{7.63}$$

Since c_i is always negative, Imag($w^{(2)}$) must be negative for A to be defined. If Imag($w^{(2)}$) is positive, there is no steady state solution. The flow is either stable with respect to any second order disturbance, or else any such disturbance leads to turbulence.

If A is defined, it gives the transition level below which the disturbance is damped out (infinitesimal

case) and above which it grows (instability).

Some idea of the physical significance of A can be seen by considering the center line fluctuations. We know that

$$(7.64)\quad v(A,y,\theta) = -\frac{\partial\psi}{\partial\theta} = -i\psi^{(1)}e^{i\theta} + i\tilde{\psi}^{(1)}e^{-i\theta} -2i\psi^{(2)}e^{2i\theta} + 2i\tilde{\psi}^{(2)}e^{-2i\theta} - 3i\psi^{(3)}e^{3i\theta} + 3i\tilde{\psi}^{(3)}e^{-3i\theta}$$

Since A is small, as a first approximation we can ignore any power of A larger than the first. Then (7.64) becomes

$$(7.65)\quad v(A,y,\theta) = -iA\,\phi^{(1,1)}(y)e^{i\theta} + iA\,\tilde{\phi}^{(1,1)}(y)e^{-i\theta}$$

or

$$(7.66)\quad v(A,y,\theta) = 2A\sin\theta(\mathrm{Real}(\phi^{(1,1)})) + 2A\cos\theta(\mathrm{Imag}(\phi^{(1,1)}))$$

Since we choose a normalization $\phi^{(1,1)}(0) = 1$, we get

$$(7.67)\quad v(A,0,\theta) = 2A\sin\theta$$

So A is approximately half the amplitude of the vertical center line disturbance.

8. METHOD OF SOLUTION

As in the linear case, we approximate the various functions by a sum of Chebyshev polynomials. The linear eigenvalue $c = -\omega^{(0)}/\alpha$ and the linear eigenfunction $\phi^{(1;1)}$ are computed using the generalized Rayleigh quotient iteration.

To find the adjoint eigenfunction v, we use the same inverse iteration, except the eigenvalue is now kept fixed. Equation (7.54) is integrated four times to obtain

$$(8.1)\quad i v - 2i\alpha^2 \iint v + i\alpha^4 \iiiint v + \alpha R \iint y v$$
$$- \alpha^3 R \iiiint y v - \alpha R c \iint v + \alpha^3 R c \iiiint v = 0$$

Letting $v(y) = \sum_{n=0}^{N} a_n T_n(y)$ and substituting into (8.1) we get the system of equations

$$(8.2)\quad i a_n - 2i\alpha^2 a_n^2 + i\alpha^4 a_n^4 + \alpha R b_n^2$$
$$- \alpha^3 R b_n^4 - \alpha R c a_n^2 + \alpha^3 R c a_n^4 = 0$$

$n = 4, 5, \ldots$

The only difference between this system and the system for the Orr-Sommerfeld equation (3.12) is that the term $-2\alpha R a_n^3$ is missing. After making this change, v(y) can be computed using the same procedure as for $\phi^{(1;1)}$.

To solve the other equations, we need subroutines for taking integrals and derivatives and for multiplying. The formula for integrals has already been worked out. If we let

$$\int \sum_{m=0}^{N} a_m T_m(y) = \sum_{m=0}^{N+1} a'_m T_m(y)$$

then we can find that

(8.3) $$a^1_n = \frac{a_{m-1} C_{m-1} - a_{m+1}}{2m} \qquad n = 1,2,\ldots,N+1$$

where a^1_0 is an arbitrary constant of integration. This is just relation (3.2)

To do derivatives, we introduce the notation

$$D\left[\sum_{m=0}^{N} a_m T_m(y)\right] = \sum_{m=0}^{N-1} a^d_m T_m(y)$$

Then by relation (3.1)

$$D\left[\sum_{m=0}^{N} a_m T_m(y)\right] = D\left[\sum_{m=0}^{N} \frac{a^d_m}{2}\left(\frac{C_m}{m+1} T_{m+1}(y) - \frac{d_{m-2}}{m-1} T_{m-1}(y)\right)\right]$$

By equating coefficients, we arrive at

$$C_{m-1} a^d_{m-1} = 2m a_m + a^d_{m+1} d_{m-1}$$

or iteratively

(8.4) $$C_m a^d_m = \sum_{k=m+1}^{\infty} 2k a_k$$

where $a_k = 0$ if $k > N$.

Multiplication can be done using the relation

(8.5) $$T_r(y) T_s(y) = \frac{1}{2}\left[T_{r+s}(y) + T_{|r-s|}(y)\right]$$

If $\phi(y) = \sum_{n=0}^{N} a_n T_n(y)$ and $\psi(y) = \sum_{n=0}^{N} b_n T_n(y)$, then

$$\phi(y)\psi(y) = \sum_{n=0}^{2N} d_n T_n(y)$$
$$= \sum_{i=0}^{N} \sum_{j=0}^{N} a_i b_j T_i(y) T_j(y)$$
$$= \sum_{i=0}^{N} \sum_{j=0}^{N} \frac{a_i b_j}{2} \left[T_{i+j}(y) + T_{|i-j|}(y) \right]$$

Truncating this sum, it can be written as

$$d_n = \sum_{k=0}^{m} \frac{a_k b_{m-k}}{2} + \frac{1}{c_m} \sum_{k=m}^{N} \frac{a_k b_{k-m}}{2} + \frac{1}{c_m} \sum_{k=0}^{N-m} \frac{a_k b_{k+m}}{2} \tag{8.6}$$

Now we can solve equation (7.52) for $\phi^{(2;2)}$. The left side is just the Orr-Sommerfeld operator with α replaced by 2α. The equation is integrated four times, and the matrix corresponding to the left side can be computed using the same subroutine as for the Orr-Sommerfeld equation.

The right side can be integrated explicitly once to obtain

$$iR\left[(D\phi^{(1;1)})^2 - \phi^{(1;1)} D^2 \phi^{(1;1)} \right] \tag{8.7}$$

Using our subroutines, the corresponding vector can be found and integrated three times. The first four coefficients are set to zero, since these equations determine the boundary conditions. The result is a system of inhomogeneous linear equations, which can be solved to obtain $\phi^{(2;2)}$.

Equation (7.51) can be explicitly integrated twice, and rewritten as

(8.8) $$D^2 \phi^{(0;2)} = \frac{R}{2}\left[-i\,\phi^{(1;1)} D \tilde{\phi}^{(1;1)} + i\,\tilde{\phi}^{(1;1)} D \phi^{(1;1)}\right]$$

The vector corresponding to the right side can be determined and then integrated twice. This gives us $\phi^{(0;2)}$ except for the first four coefficients, which are determined from the boundary conditions.

Equation (7.60) can now be solved for $w^{(2)}$. The definite integrals can be evaluated using (3.1) and the formula

(8.9) $T_n(\pm 1) = (\pm 1)^m$

These allow us to compute

$$\int_{-1}^{1} \sum_{m=0}^{N} a_m T_m(y)\,dy = \sum_{m=0}^{N} \frac{a_m}{2}\left[\frac{c_m T_{m+1}(y)}{m+1} - \frac{d_{m-2} T_{m-1}(y)}{m-1}\right]_{-1}^{1}$$

$$= \sum_{m=0}^{N} \frac{a_m}{2}\left[\frac{c_m}{m+1}\left(1-(-1)^{m+1}\right) - \frac{d_{m-2}}{m-1}\left(1-(-1)^{m-1}\right)\right]$$

$$= \sum_{m=0}^{N,2} a_m\left[\frac{c_m}{m+1} - \frac{d_{m-2}}{m-1}\right]$$

$$= 2a_0 + \sum_{m=2}^{N,2}\left[\frac{1}{m+1} - \frac{1}{m-1}\right] a_m$$

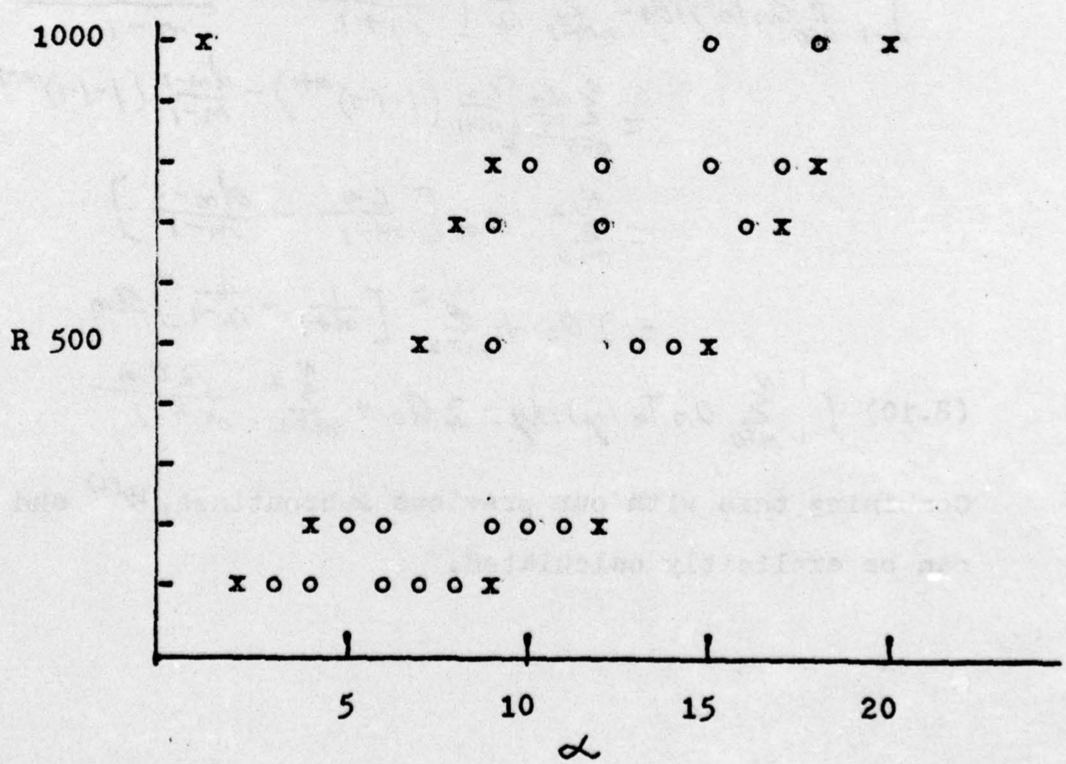
(8.10) $$\int_{-1}^{1} \sum_{m=0}^{N} a_m T_m(y)\,dy = 2a_0 + \sum_{m=2}^{N,2} \frac{-2a_m}{m^2-1}$$

Combining this with our previous subroutines, $w^{(2)}$ and A can be explicitly calculated.

9. NUMERICAL RESULTS

The above set of equations was solved for a variety of values of α and R. The results are summarized in Figure 2, where an o represents values of the parameters where a steady state solution exists, and an x values for which there is no such solution.

Fig. 2.--Location of steady state solutions



The size of the disturbances associated with the steady state solutions varies widely. Table 7 shows the

amplitude A of the steady state solutions for different values of α and R.

TABLE 7

VALUES OF R, α, AND A

R	α	A	R	α	A
100	3	1.57×10^{-1}	500	13	4.60×10^{-6}
100	4	2.44×10^{-2}	500	14	2.61×10^{-6}
100	6	7.02×10^{-3}	700	9	1.76×10^{-4}
100	7	4.33×10^{-3}	700	12	6.10×10^{-6}
100	8	3.20×10^{-3}	700	16	1.18×10^{-6}
200	5	7.59×10^{-3}	800	10	4.83×10^{-5}
200	6	2.85×10^{-3}	800	12	5.91×10^{-6}
200	9	3.51×10^{-4}	800	15	8.40×10^{-7}
200	10	2.19×10^{-4}	800	17	7.02×10^{-7}
200	11	4.12×10^{-4}	1000	15	5.65×10^{-7}
500	9	9.83×10^{-5}	1000	18	2.45×10^{-7}

As in experiments, the smaller the Reynolds number R, the larger the corresponding amplitude. The smallest value computed was R = 5 and α = 2. The method predicted a steady state solution, but with an amplitude of 6.05. Since our formalism assumed small disturbances, this does not reflect a realistic solution.

It was not feasible to determine whether there exists

a largest value of R for which there is a steady state solution, because of the computational expense. To obtain accurate eigenfunctions proved more difficult than to obtain accurate eigenvalues, particularly if both α and R were relatively large.

But steady state solutions are possible at least up to R = 1000. This would imply that small disturbances will die out. However, in practice, laminar flow can be maintained only up to about R = 750. Since in an actual apparatus there are always some perturbations, this could be due to the fact that these perturbations can no longer be reduced below the transition level. Other effects not covered by our formalism could also be at work. However, just considering second order effects, this method does correctly predict transition to turbulence for a large enough perturbation, which the linear theory does not.

Some insights into the results can be obtained by considering the shape of the disturbances. Using a second order approximation

$$(9.1)\quad \psi(A,y,\theta) = 2\psi^{(0)} + \psi^{(1)}e^{i\theta} + \tilde{\psi}^{(1)}e^{-i\theta} + \psi^{(2)}e^{2i\theta} + \tilde{\psi}^{(2)}e^{-2i\theta}$$

or

$$(9.2)\quad \psi(A,y,\theta) = \frac{\alpha}{2}y^2 - \frac{\alpha}{2} + 2A\left[\mathrm{Real}(\phi^{(1;1)})\cos\theta - \mathrm{Imag}(\phi^{(1;1)})\sin\theta\right] + 2A^2\left[\phi^{(0;2)} + \mathrm{Real}(\phi^{(2;2)})\cos 2\theta - \mathrm{Imag}(\phi^{(2;2)})\sin 2\theta\right]$$

($\phi^{(0;2)}$ is a real valued function.) Then since $u = \frac{1}{\alpha}\frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial\theta}$.

$$(9.3)\quad u(A,y,\theta) = y + \frac{2}{\alpha}A\left[D(\text{Real}\,\phi^{(1;1)})\cos\theta - D(\text{Imag}\,\phi^{(1;1)})\sin\theta\right] + \frac{2}{\alpha}A^2\left[D\phi^{(0;2)} + D(\text{Real}\,\phi^{(2;2)})\cos 2\theta - D(\text{Imag}\,\phi^{(2;2)})\sin 2\theta\right]$$

$$(9.4)\quad v(A,y,\theta) = -2A\left[(\text{Real}\,\phi^{(1;1)})\sin\theta + (\text{Imag}\,\phi^{(1;1)})\cos\theta\right] - 4A^2\left[(\text{Real}\,\phi^{(2;2)})\sin 2\theta + (\text{Imag}\,\phi^{(2;2)})\cos 2\theta\right]$$

If we ignore the A^2 terms, we have the Orr-Sommerfeld linear solution.

Choosing $\theta = 0$, this simplifies to

$$(9.5)\quad u(A,y,0) = y + \frac{2}{\alpha}A\,D(\text{Real}\,\phi^{(1;1)}) + \frac{2}{\alpha}A^2\left[D\phi^{(0;2)} + D(\text{Real}\,\phi^{(2;2)})\right]$$

$$(9.6)\quad v(A,y,0) = -2A\,\text{Imag}\,\phi^{(1;1)} - 4A^2\,\text{Imag}\,\phi^{(2;2)}$$

Using the definition $T_n(\cos\gamma) = \cos n\gamma$, these vectors can be computed. This is done most easily by choosing values of γ between 0 and π. Then $y = \cos\gamma$ and $T_n(y) = \cos n\gamma$.

The resulting solutions are graphed for the case R = 200 and $\alpha = 5$. Figure 3 gives u and v for the linear case, and Figure 4 gives the non-linear solutions.

For the linear case, the disturbance is concentrated near the boundary y = 1. There is of course a symmetric

Fig. 3.--The linear solution for R = 200 and α = 5

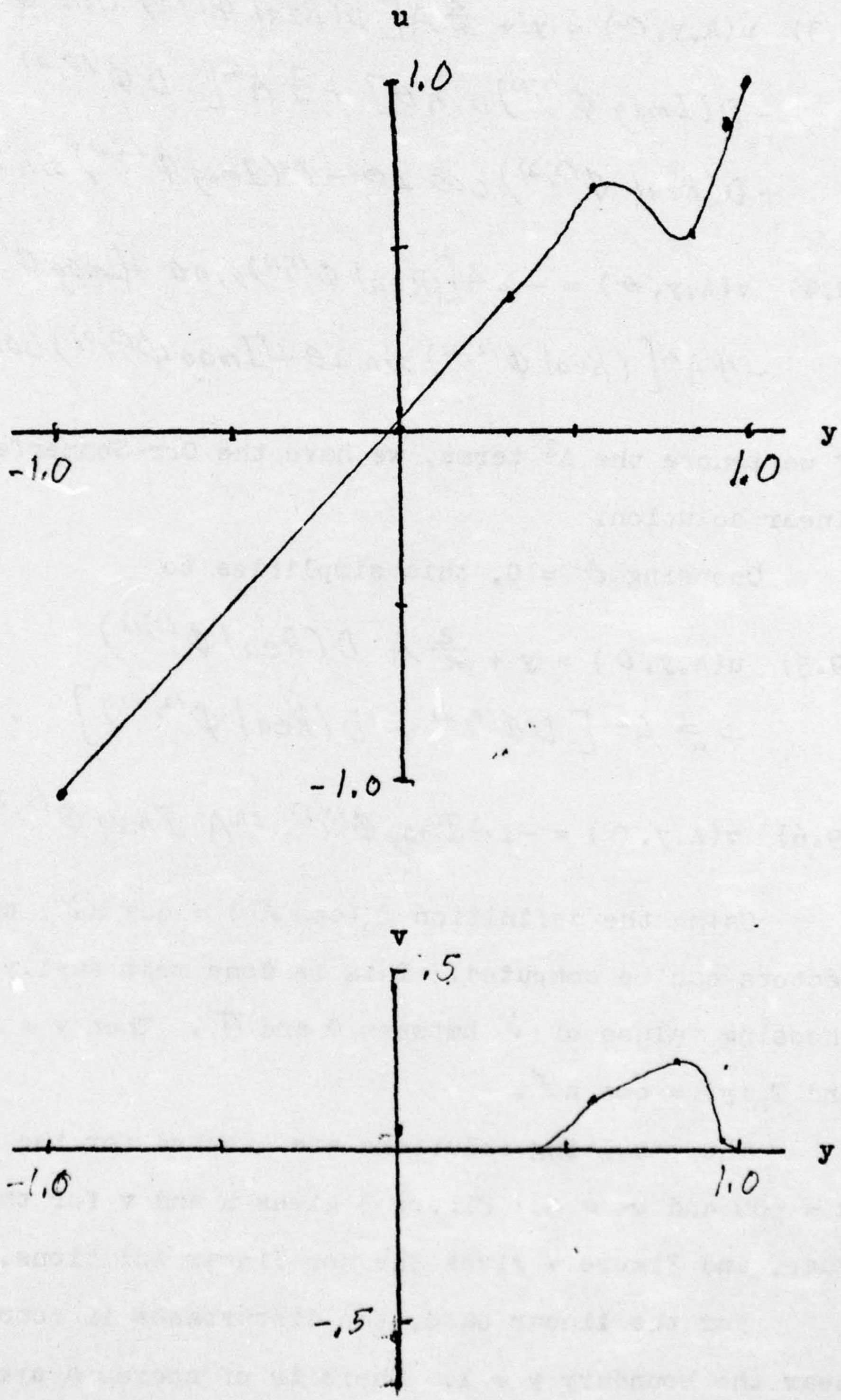
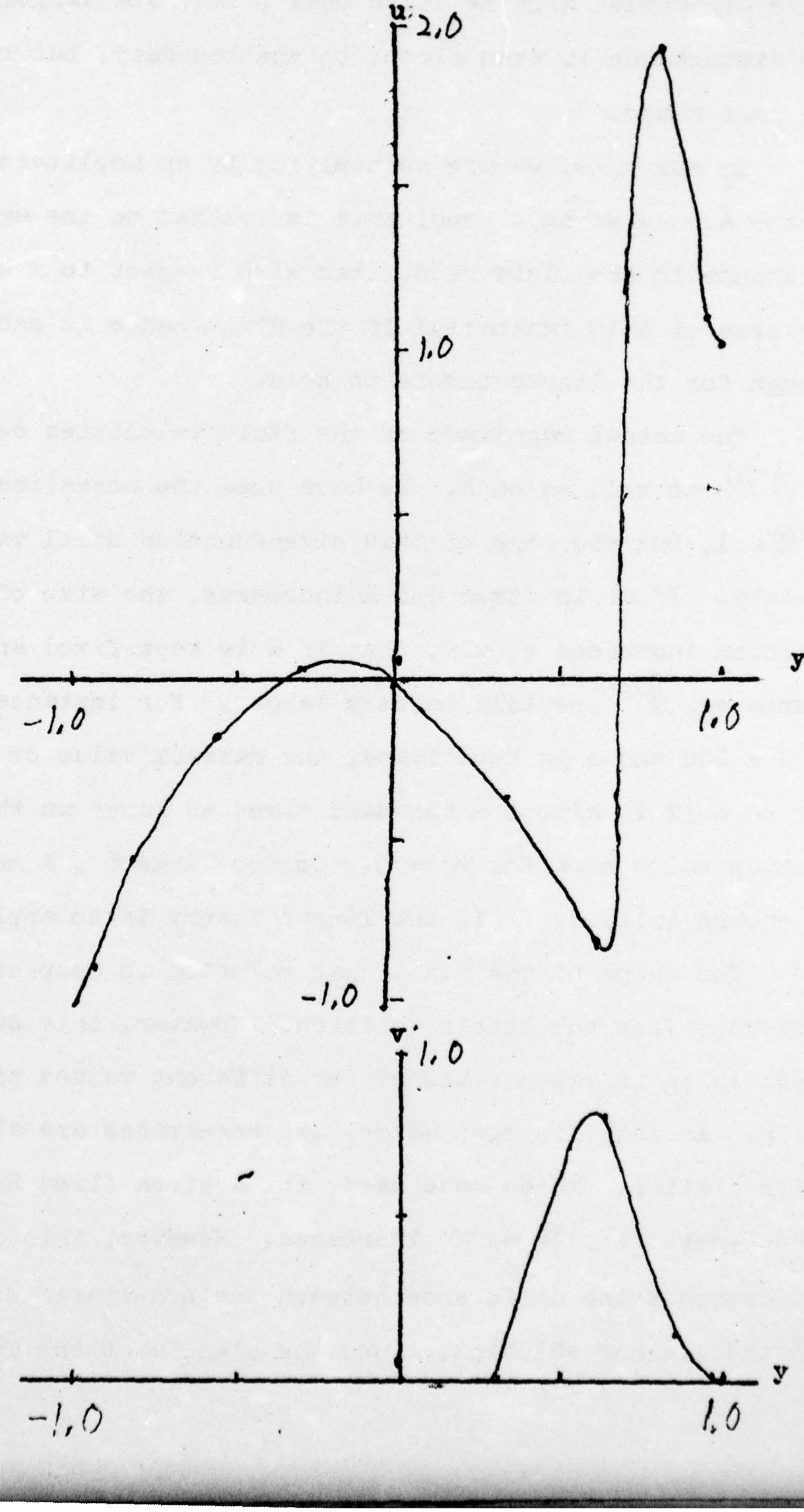


Fig. 4.--The non-linear solution for R = 200 and α = 5



solution concentrated near the boundary $y = -1$, corresponding to an eigenvalue with negative real part. For larger R, the disturbance is even closer to the boundary, but retains the same shape.

In our case, we are multiplying by an amplitude factor A. As we have seen, this is related to the amount of change in the fluid velocities with respect to x and t. The size of A is immaterial if the disturbance is small enough for the linear theory to hold.

The actual magnitude of the fluid velocities depends on $\phi^{(1,1)}$ as well as on A. We have used the normalization $\phi^{(1,1)}(0) = 1$, but the size of this eigenfunction still varies greatly. If α is fixed and R increases, the size of this function increases slowly. But if R is kept fixed and α increases, $\phi^{(1,1)}$ rapidly becomes larger. For instance, if $R = 200$ and A is kept fixed, the maximum value of u for $\alpha = 12$ is almost a thousand times as large as the maximum value of u for $\alpha = 5$. So for large α, A must be chosen quite small if the linear theory is to apply.

The shape of the non-linear solution is completely different from the linear solution. However, this same basic shape is again retained for different values of α and R. In fact, in most cases, the magnitudes are also quite similar. As we have seen, for a given fixed R, A decreases rapidly as α increases. However, this does not decrease the difference between the non-linear solution and the laminar solution, since the eigenfunctions are

getting large rapidly.

The value of A is now important, since the given second order solution only exists for the appropriate amplitude. When we multiply by A, as we have been doing, the linear solution becomes closer and closer to the laminar solution as R increases. But the second order solutions remain distinct from the laminar case.

The transition from the linear to the second order solution cannot be obtained by our method. However, we know that for small disturbances the linear theory holds. Somewhere well before the transition amplitude non-linear effects would begin to occur. For these intermediate disturbances the shape would begin to change from the linear to the non-linear form. By the time the transition amplitude is reached, the non-linear effects have drastically changed the shape and the size of the solution. So our theory shows a large difference between the linear and non-linear solution, which explains how the transition to turbulence can occur.

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